Group Theory 6

07 October 2023 11:02

- If $S \leqslant G$ then S is a group in its own.
- A subset S of a group G is a subgroup if and only if $1 \in S$ and $s, t \in S \Rightarrow st^{-1} \in S$
- <u>Def</u>:- If G is a group and $a \in G$ and the cyclic group generated by a is denoted as $\langle a \rangle$ is set of all powers of a
  G is called cyclic if $\exists\, a \in G$ such that $G = \langle a \rangle$

Order of $a \in G$, denoted by $\mathrm{Ord}(a)$ is $|\langle a \rangle|$ = number of elements of $\langle a \rangle$
$= \min(m)$ such that $a^m = 1$ and $m \in \mathbb{Z}^+$

- G is a finite group and S is a non-empty subset of G. Then S is a subgroup if and only if $s, t \in S \Rightarrow st \in S$

- <u>Definition</u>:- $f : G \to H$ is a homomorphism
  $\mathrm{Kernel}(f) = \{a \in G,\ f(a) = 1\}$
  $\mathrm{image}(f) = \{h \in H,\ h = f(a) \text{ for some } a \in G\}$

- $\ker(f)$ is a subgroup of G and $\mathrm{img}(f)$ is a subgroup of H

<u>Theorem</u>:- Intersection of any family of subgroups of G is a subgroup of G.

<u>Proof</u>:- $F = \{H_i : H_i \leqslant G\} \Rightarrow \bigcap_i H_i$ contains 1
$a, b \in \bigcap_i H_i \Longrightarrow a, b \in H_i\ \forall i \Rightarrow ab^{-1} \in H_i\ \forall i \Rightarrow ab^{-1} \in \bigcap_i H_i$
So, $\bigcap_i H_i \leqslant G$

Corollary:- If X is a subset of a group G, then there is a

Corollary:– If X is a subset of a group G, then there is a smallest subgroup H of G containing X, i.e., if $X \subset S$ and $S \leq G$ then $H \leq S$

Definition:– X is non-empty subset of a group G, then a word on X is an element $w \in G$ of the form

$$w = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$$

where $x_i \in X$ and $e_i = \pm 1$ and $n \geq 1$.
(x_i, x_j may be same)

Theorem:– X be a subset of a group G. If $X = \phi$ then $\langle X \rangle = 1$, if X is non-empty, then $\langle X \rangle$ is set of all words on X

Proof:– If $X = \phi$,
X generates $\langle X \rangle$. $\Rightarrow$ $\langle X \rangle = 1$.

If $X \neq \phi$,
W be the word of X
$1 \in W$ as $x_i^{-1} x_i \in W$, $x_i^{-1} \in W$, $\forall x_i \in X$
$x_i \in W$ $\forall x_i \in X$

$\langle X \rangle \subset W$ as $\langle X \rangle$ is the smallest subgroup.

arbitrary,
$a \in W \Rightarrow a = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$
$\Rightarrow x_1^{e_1} \in \langle X \rangle, x_2^{e_2} \in \langle X \rangle, \ldots, x_n^{e_n} \in \langle X \rangle$
$\Rightarrow a \in \langle X \rangle \Rightarrow W \subset \langle X \rangle$

$W \subset \langle X \rangle$ & $\langle X \rangle \subset W \Rightarrow W = \langle X \rangle$

Q) S be a proper subgroup of G. If $G-S$ is the complement of S in G, then prove that $\langle G-S \rangle = G$

Ans:– $x \in \langle G-S \rangle$ $\quad x = g_1^{e_1} g_2^{e_2} \cdots g_r^{e_r}$ $\quad g_i \in G$
$x \in G$

Ans:— $x \in \langle G-S \rangle$ $\quad x = y_1 \, y_2 \cdots y_r$

$x \in G$

$x \in (G-S)$ then $x \in \langle G-S \rangle$

$x \in S$, $\exists \, y \in (G-S)$ such that $xy \notin S$, $xy \in (G-S)$

$xy \in \langle G-S \rangle$

$y^{-1} \in (G-S)$ $\quad xyy^{-1} \in \langle G-S \rangle \Rightarrow x \in \langle G-S \rangle$

$\langle G-S \rangle \supset S$ & $\langle G-S \rangle \supset (G-S)$

$\Rightarrow \langle G-S \rangle = G$

Q> Generators of S_n are what?

Ans:— $(1,2), (1,3) \ldots\ldots, (1,n)$

$(r_1 \, r_2 \, r_3 \cdot \; r_n)$

$= (r_1 \, r_n)(r_1 \, r_{n-1}) \cdots (r_1 \, r_2)$

$= (1 \, r_1)(1 \, r_n)(1 \, r_1) \ldots$

Q> Prove that S_4 cannot be generated by $(1\,3)$ and $(1\,2\,3\,4)$

Ans:— $(1\,2\,3\,4) = (1\,4)(1\,3)(1\,2)$; $\quad (1\,3)$

$(1234)^4 = e$, $\quad (13)^2 = e$

$(1234)(13) = (14)(23) = (13)(4\,3\,2\,1)$

$= (13)(1\,2\,3\,4)^{-1}$

$\langle (1,3), (1234) \rangle = D_4 \qquad |D_4| = 2 \times 4 = 8$

$|S_4| = 24 = 4!$ $\qquad$ So $\langle (1,3), (1234) \rangle \neq S_4$

Q: Let G be finite group. Prove that the number of elements of x of G such that $x^7 = e$ is odd

Ans:— if $x \neq e$, $\mathrm{Ord}(x) = 7$ as 7 is a prime

$(x^i)^7 = e \;\; \forall \; 1 \leq i \leq 6$

... multiple of 6

$$(x^i)^7 = e \ \forall \ 1 \le i \le 6$$

No. of elements of this form is multiple of 6

And we have $e^7 = e$

So total no. of elements $= 6n+1$ which is odd.